a symmetric matrix.

We see that the plastic deformation process is performed without a change in volume or $\Delta_p = \det A_p = 1$.

Indeed, by definition

$$\Delta_p^{\cdot} = A_{pij}^{\cdot}\Delta_{pij} = \Delta_p A_{pij}^{\cdot} A_{pji}^{-1} = \Delta_p A_e A_e^* : \varphi = -\Delta_p G_e^{-1} : \varphi$$

The condition $\Delta_p^{\cdot} = 0$ is equivalent to the requirement of satisfying the continuity equation /1/, as is hence seen.

Now, when the equivalence of the equations obtained to the system of equations in /1/ is established, the results of /6, 7/ can be used for their closure, where semi-empirical equations of state and interpolation formulas of the kinetics of plastic deformation for a number of metals are presented.

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ASYMPTOTIC SOLUTION OF A QUASISTATIC THERMOELASTICITY PROBLEM FOR A SLENDER ROD*

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An asymptotic expansion is constructed for solving a quasistatic thermoelasticity problem for a slender cylindrical rod in the presence of mass forces and non-linear heat sources. The algorithm for constructing the asymptotic form, based on the method of boundary functions, is fairly simple and convenient for carrying out numerical calculations. A deduction is made on the basis of the asymptotic form constructed on how to select correctly a simplified one-dimensional model so as to obtain a better approximation for the solution of the initial two-dimensional problem. An existence theorem for the solution is proved under certain conditions.

1. Formulation of the problem.

In the linear approximation the system of thermoelasticity equations for the displacement vector $\mathbf{u}(x, y, z, t)$ and temperature $\theta(x, y, z, t)$ in a certain domain G has the form /1/

$$\mu\Delta\mathbf{u} + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + \mathbf{X} = \gamma\,\mathrm{grad}\,\theta + \rho_0\mathbf{u}^{\cdot\cdot} \qquad (1.1)$$
$$\Delta\theta - \varkappa^{-1}\theta^{\cdot} - \eta\,\mathrm{div}\,\mathbf{u}^{\cdot} = -\varkappa^{-1}H$$

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$$\gamma = (3\lambda + 2\mu)\,\alpha,\quad \varkappa = \lambda_0/c,\quad \eta = \gamma\,\langle\theta\rangle/\lambda_0$$

The dot here denotes the partial derivatives with respect to time, μ, λ are the elastic moduli (quantities characterizing the elastic properties of the materials under small strains), $\mathbf{X}(x, y, z, t)$ is the mass force vector in the domain, α is the coefficient of linear thermal expansion, $\rho_0(x, y, z)$ is the volume density in the domain G, λ_0 is the thermal conductivity, c is the specific heat under constant strain, $\langle\theta\rangle$ is the mean body temperature, and $H(\theta, x, y, z, t)$ are thermal sources in the domain. System (1.1) is written under the following assumptions: the change in θ is small, and does not result in changes in the thermal and elastic constants, and the relationships between the displacements and strains are linear.

We consider a thermoelasticity boundary value problem for a slender rod of radius εb and length a ($\varepsilon > 0$ is a small parameter). To do this we will change to cylindrical coordinates ρ, φ, z and we will seek the axisymmetric solution, i.e., independent of φ. Then system (1.1) will take the form

$$\begin{aligned}
&\mu\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial v}{\partial\rho}\right) + \frac{\partial^2 v}{\partial z^2} - \frac{v}{\rho^2}\right) + \\
&\quad(\lambda+\mu)\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho v) + \frac{\partial w}{\partial z}\right) + F = \gamma\frac{\partial\theta}{\partial\rho} + \rho_0 v^{\cdot\cdot} \\
&\mu\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial w}{\partial\rho}\right) + \frac{\partial^2 w}{\partial z^2}\right) + (\lambda+\mu)\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho v) + \frac{\partial w}{\partial z}\right) + f = \\
&\quad\gamma\frac{\partial\theta}{\partial z} + \rho_0 w^{\cdot\cdot} \\
&\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\theta}{\partial\rho}\right) + \frac{\partial^2\theta}{\partial z^2} - \varkappa^{-1}\theta^{\cdot} - \eta\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho v^{\cdot}) + \frac{\partial w^{\cdot}}{\partial z}\right) = -\varkappa^{-1}H
\end{aligned} \tag{1.2}$$

where $v(\rho, z, t)$ and $w(\rho, z, t)$ are the radial and axial displacements, and $F(\rho, z, t)$ and $f(\rho, z, t)$ are the radial and axial components of the vector $\mathbf{X}(\rho, z, t)$.

We impose the following additional conditions.

On the side surface of the rod

$$\rho = \varepsilon b,\quad v = \varepsilon\varphi(z, t, \varepsilon),\quad \frac{\partial w}{\partial\rho} = 0,\quad \frac{\partial\theta}{\partial\rho} + A\varepsilon\theta = 0$$

The first condition shows that with the passage of time, points of the rod side surface will undergo small radial displacements in conformity with the function $\varepsilon\varphi(z, t, \varepsilon)$, in particular rigidly clamped if $\varphi(z, t, \varepsilon) = 0$. The second condition means that the layers abutting the side surface are homogeneous in the radial direction. And, finally the third condition shows that weak heat transfer (of the order of ε) to the environment occurs according to Newton's law on the rod side surface.

At the ends of the rod

$$z = 0,\quad \frac{\partial w}{\partial\rho} + \frac{\partial v}{\partial z} = 0,\quad w = \psi_1(\rho, t),\quad \theta = \mu_1(\rho, t)$$

$$z = a,\quad \frac{\partial w}{\partial\rho} + \frac{\partial v}{\partial z} = 0,\quad w = \psi_2(\rho, t),\quad \theta = \mu_2(\rho, t)$$

The mixed-type conditions indicate the absence of shear stresses, and the remaining conditions yield changes in the axial displacements and temperatures at the ends of the rod with time.

The initial condition for the temperature is

$$t = 0,\quad \theta = \chi(\rho, z)$$

It should also be necessary to give initial conditions for v and w for system (1.2). But we shall later examine a shortened system. It is known /1/ that if the mass forces, surface forces, and thermal sources vary slowly with time, then the components $\rho_0 v^{\cdot\cdot}, \rho_0 w^{\cdot\cdot}$ in system (1.2) can be neglected and the so-called quasistatic thermoelasticity problem can be solved. We shall indeed consider just such a problem. We take into account here that the ratio $\gamma/(\lambda + 2\mu)$ is of the order of 10^{-6}–$10^{-5}\,K^{-1}$ for a broad class of substances and we set $\gamma = \varepsilon\beta$.

Let us make the change of variable $\rho = \varepsilon r$. Then the problem for $y = (v, w, \theta)$ takes the form of a singularly perturbed problem in the variables r, z, t (the small parameter ε enters as a factor in the derivatives)

$$\begin{aligned}
&L_1 y \equiv (\lambda + 2\mu)\left(L_0 v - \frac{v}{r^2}\right) + \varepsilon^2\mu\frac{\partial^2 v}{\partial z^2} + \varepsilon(\lambda+\mu)\frac{\partial^2 w}{\partial r\partial z} = \\
&\quad -\varepsilon^2 F + \varepsilon^2\beta\frac{\partial\theta}{\partial r} \\
&L_2 y \equiv \mu L_0 w + \varepsilon^2(\lambda+2\mu)\frac{\partial^2 w}{\partial z^2} + \varepsilon(\lambda+\mu)\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv)\right) = -\varepsilon^2 f + \varepsilon^3\beta\frac{\partial\theta}{\partial z}
\end{aligned} \tag{1.3}$$

$$L_3y \equiv L_0\theta + \varepsilon^2 \frac{\partial^2\theta}{\partial z^2} - \varepsilon^2\varkappa^{-1}\theta^{\cdot} - \varepsilon\eta \frac{1}{r}\frac{\partial}{\partial r}(rv^{\cdot}) - \varepsilon^2\eta\frac{\partial w^{\cdot}}{\partial z} = -\varepsilon^2\varkappa^{-1}H$$
$$L_0 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}; \quad (r, z, t) \in G = (0 \leqslant r < b,\ 0 < z < a,\ 0 < t \leqslant T)$$
$$r = b, \quad v = \varepsilon\varphi(z, t, \varepsilon), \quad \frac{\partial w}{\partial r} = 0, \quad \frac{\partial\theta}{\partial r} + A\varepsilon^2\theta = 0 \tag{1.4}$$
$$z = 0, \quad \frac{\partial w}{\partial r} + \varepsilon\frac{\partial v}{\partial z} = 0, \quad w = \psi_1(r, t, \varepsilon), \quad \theta = \mu_1(r, t, \varepsilon)$$
$$z = a, \quad \frac{\partial w}{\partial r} + \varepsilon\frac{\partial v}{\partial z} = 0, \quad w = \psi_2(r, t, \varepsilon), \quad \theta = \mu_2(r, t, \varepsilon)$$
$$t = 0, \quad \theta = \chi(r, z, \varepsilon)$$

(We consider the dependence of the functions ψ_i, μ_i $(i = 1, 2)$, χ, F, f, H on r and ε in the boundary and initial conditions and the inhomogeneities of the equations and not on $r\varepsilon$ as is obtained after the substitution $\rho = \varepsilon r$.)

Note that the stationary thermoelasticity problem (the term $\theta^{\cdot}$ is also discarded) for a slender rod with other boundary conditions was considered in /2/ in the case $X \equiv 0, H \equiv 0$.

The purpose of this paper is to construct an asymptotic solution of problem (1.3) and (1.4) under the following requirements.

1°. All the known functions in (1.3) and (1.4) are sufficiently smooth.

2°. Conditions for matching the boundary values for v and w and also the initial and boundary values for θ are satisfied:

$$\frac{\partial\psi_i(b, t, \varepsilon)}{\partial r} = 0, \qquad \frac{\partial\varphi(0, t, \varepsilon)}{\partial z} = \frac{\partial\varphi(a, t, \varepsilon)}{\partial z} = 0$$
$$\mu_1(r, 0, \varepsilon) = \chi(r, 0, \varepsilon), \ \mu_2(r, 0, \varepsilon) = \chi(r, a, \varepsilon)$$
$$\frac{\partial\mu_i(b, t, \varepsilon)}{\partial r} = 0, \ \frac{\partial\chi(b, t, \varepsilon)}{\partial r} = 0, \quad i = 1, 2$$
$$\frac{\partial\psi_i(0, t, \varepsilon)}{\partial r} = \frac{\partial\mu_i(0, t, \varepsilon)}{\partial r} = \frac{\partial\chi(0, z, \varepsilon)}{\partial r} = 0, \quad i = 1, 2$$

3°. The last requirement is the necessary condition for a smooth axisymmetric solution to exist.

The remaining requirements will be imposed during the construction of the asymptotic solution.

2. An algorithm for constructing the asympotic form.

We will seek the asymptotic form for the solution of problem (1.3) and (1.4) in a form characteristic for the method of boundary functions (BF) /3/

$$y(r, z, t, \varepsilon) = \bar{y}(r, z, t, \varepsilon) + Qy(r, \xi, t, \varepsilon) + Q^*y(r, \xi_*, t, \varepsilon) + \Pi y(r, z, \tau, \varepsilon) + Py(r, \xi, \tau, \varepsilon) + P^*y(r, \xi_*, \tau, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i (\bar{y}_i(r, z, t) + Q_iy(r, \xi, t) + Q_i^*y(r, \xi_*, t) + \Pi_iy(r, z, \tau) + P_iy(r, \xi, \tau) + P_i^*y(r, \xi_*, \tau)); \tag{2.1}$$
$$\xi = \frac{z}{\varepsilon}, \quad \xi_* = \frac{a - z}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon^2}$$

Here $\bar{y}$ is the regular part of the asymptotic form, $Qy, \ldots, P^*y$ are boundary functions whose purpose is described below, and ξ, ξ_*, τ are boundary layer variables.

Substituting (2.1) into (1.3) and using the representation /4/ $H = \bar{H} + QH + Q^*H + \Pi H + PH + P^*H$, we obtain an equation for the terms of the asymptotic form by a standard method (series expansions in powers of ε). The functions $F, f, \varphi, \psi_i, \mu_i$ $(i = 1, 2)$, χ are also represented in the form of series in powers of ε, for instance

$$F(r, z, t, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i F_i(r, z, t)$$

2.1. The regular part of the asymptotic form. For $\bar{v}_0(r, z, t)$ we obtain the problem

$$L_0\bar{v}_0 - r^{-2}\bar{v}_0 = 0; \ r = b, \ \bar{v}_0 = 0$$

whose solution, bounded in $\bar{G}$ is $\bar{v}_0 = 0$.

The function $\bar{w}_0(r, z, t)$ is determined from the system

$$L_0\bar{w}_0 = 0; \quad r = b, \quad \partial\bar{w}_0/\partial r = 0 \tag{2.2}$$

Its solution is an arbitrary function of the variables $z, t: \bar{w}_0 = g_0(z, t)$. For $\theta_0(r, z, t)$ we obtain a problem analogous to (2.2). Consequently, $\bar{\theta}_0 = \alpha_0(z, t)$ is an arbitrary function of z and t.

Therefore, problem (1.3) and (1.4) refers to the so-called critical cases in the theory of singular perturbations: the solution of the degenerate problem $(\varepsilon = 0)$ is not determined single-valuedly, but depends on arbitrary functions /5/. (An analogous situation is considered in /6/ for the heat conduction equation in a slender rod).

For $\bar{v}_1(r, z, t)$ we obtain the problem

$$L_0\bar{v}_1 - r^{-2}\bar{v}_1 = 0; \quad r = b, \quad \bar{v}_1 = \varphi_0(z, t) \tag{2.3}$$

Its solution is $\bar{v}_1 = \varphi_0(z, t)r/b$. The functions $\bar{w}_1$ and $\bar{\theta}_1$: $\bar{w}_1 = g_1(z, t), \bar{\theta}_1 = \alpha_1(z, t)$ (g_1 and α_1 are arbitrary functions) are determined in the same way as the functions $\bar{w}_0, \bar{\theta}_0$.

For $\bar{v}_i(r, z, t)$ for $i \geqslant 2$ we obtain a problem analogous to (2.3) with a non-zero right side in the equations. Hence $\bar{v}_i$ is determined single-valuedly.

For $\bar{w}_2(r, z, t)$ we also obtain an inhomogeneous problem

$$\mu L_0\bar{w}_2 = -(\lambda + 2\mu)\frac{\partial^2 g_0}{\partial z^2} - \frac{2(\lambda+\mu)}{b}\frac{\partial\varphi_0}{\partial z} - f_0 \tag{2.4}$$
$$r = b, \ \partial\bar{w}_2/\partial r = 0$$

Note that the second derivative of the still unknown function $g_0(z, t)$ is in the right-hand side of the equation. The general solution of the equation bounded in $\bar{G}$ is

$$\bar{w}_2 = \frac{1}{4\mu}\left\{-(\lambda+2\mu)\frac{\partial^2 g_0}{\partial z^2} - \frac{2(\lambda+\mu)}{b}\frac{\partial\varphi_0}{\partial z}\right\}r^2 -$$
$$f_0^+(r, z, t) + g_2(z, t), \quad f_0^+ = \frac{1}{\mu}\int_0^r \frac{d\sigma}{\sigma}\int_0^\sigma \eta f_0(\eta, z, t)\,d\eta$$

where g_2 is an arbitrary function. This solution satisfies the boundary condition in (2.4) only if the equality is satisfied (the solvability condition for (2.4))

$$\frac{\partial^2 g_0}{\partial z^2} = -\frac{2(\lambda+\mu)}{b(\lambda+2\mu)}\frac{\partial\varphi_0}{\partial z} - \frac{2\mu}{b(\lambda+2\mu)}\frac{\partial f_0^+(b, z, t)}{\partial r} \tag{2.5}$$

Condition (2.5) is an equation in the function $g_0(z, t)$. The boundary conditions for g_0 will be obtained when the BF is constructed. By virtue of (2.5)

$$\bar{w}_2 = \frac{\partial f_0^+(b, z, t)}{\partial r}\frac{r^2}{2b} - f_0^+(r, z, t) + g_2(z, t)$$

The function $\bar{\theta}_2(r, z, t)$ is a solution of the problem

$$L_0\bar{\theta}_2 = -\frac{\partial\alpha_0}{\partial z^2} + \varkappa^{-1}\alpha_0^{\cdot} + \eta\left(\frac{2}{b}\varphi_0^{\cdot} + \frac{\partial g_0^{\cdot}}{\partial z}\right) - \tag{2.6}$$
$$\varkappa^{-1}H_0(\alpha_0, r, z, t); \quad r = b, \quad \partial\bar{\theta}_2/\partial r = -A\alpha_0(z, t)$$

Derivatives of the still unknown functions $\alpha_0(z, t)$ and $g_0(z, t)$ are on the right-hand side of the equation. The solution (2.6) is determined apart from the arbitrary function $\alpha_2(z, t)$

$$\bar{\theta}_2 = \left\{-A\alpha_0 + \frac{\partial H_0^+(\alpha_0, b, z, t)}{\partial r}\right\}\frac{r^2}{2b} - H_0^+(\alpha_0, r, z, t) + \alpha_2$$
$$H_0^+ = \frac{1}{\varkappa}\int_0^r \frac{d\sigma}{\sigma}\int_0^\sigma \eta H_0(\alpha_0, \eta, t)\,d\eta$$

while for $\alpha_0(z, t)$ we obtain a parabolic equation

$$\alpha_0^{\cdot} - \varkappa\partial^2\alpha_0/\partial z^2 = K(\alpha_0, z, t), \quad K(\alpha_0, z, t) = \tag{2.7}$$
$$\frac{2}{b}\frac{\partial H_0^+(\alpha_0, b, z, t)}{\partial r} - \eta\varkappa\left(\frac{2}{b}\varphi_0^{\cdot} + \frac{\partial g_0}{\partial z}\right) - \frac{2A}{b}\varkappa\alpha_0$$

from the condition for problem (2.6) to be solvable.

The initial and boundary conditions for the function α_0 will be obtained below when constructing the BF.

For $\bar{w}_i(r, z, t)$ and $\bar{\theta}_i(r, z, t)$ with $i \geqslant 3$ we have problems analogous to (2.4) and (2.6). From the conditions for these problems to be solvable we obtain an equation of the type (2.5) for $g_{i-2}(z, t)$ and a linear equation of the type (2.7) for $\alpha_{i-2}(z, t)$.

Therefore, the functions $\bar{v}_i$ are determined uniquely at each step and the functions $\bar{w}_i$

and $\overline{\theta}_i$, apart from the arbitrary functions $g_i(z, t)$ and $\alpha_i(z, t)$, for which equations of the types (2.5) and (2.7) are obtained from the conditions for the problems to be solvable for $\overline{w}_{i+2}$ and $\overline{\theta}_{i+2}$. The terms of the regular part of the asymptotic form will be determined uniquely after constructing the BF.

2.2. The boundary layer in the neighbourhoods of the rod ends. The function $Qy(r, \xi, t, \varepsilon)$ serves to satisfy a given boundary condition for $z = 0$ in combination with the regular part of the asymptotic form. Moreover, the function $Qy(r, \xi, t, \varepsilon)$ should be a BF in the variable ξ, i.e.,

$$Qy(r, \xi, t, \varepsilon) \to 0 \text{ as } \xi \to \infty \tag{2.8}$$

For $Q_0v(r, \xi, t)$, $Q_0w(r, \xi, t)$ we obtain the problem

$$(\lambda + 2\mu)(L_0Q_0v - r^{-2}Q_0v) + (\lambda + \mu)\frac{\partial^2 Q_0w}{\partial r\partial\xi} + \mu\frac{\partial^2 Q_0v}{\partial\xi^2} = 0$$

$$\mu L_0Q_0w + (\lambda + 2\mu)\frac{\partial^2 Q_0w}{\partial\xi^2} + (\lambda + \mu)\frac{\partial}{\partial\xi}\left(\frac{1}{r}\frac{\partial}{\partial r}(rQ_0v)\right) = 0$$

$$r = b,\ Q_0v = 0,\ \partial Q_0w/\partial r = 0$$

$$\xi = 0,\ \partial Q_0w/\partial r + \partial Q_0v/\partial\xi = 0,\ Q_0w = \psi_{10}(r, t) - g_0(0, t)$$

We note that $g_0(0, t)$, the value of the still unknown function $g_0(z, t)$ at $z = 0$, occurs in the boundary condition. We will seek the solution of this problem in the form

$$Q_0v = \sum_{n=0}^{\infty} q_{0n}(t, \xi) J_1(\nu_n r), \quad Q_0w = \sum_{n=0}^{\infty} p_{0n}(t, \xi) J_0(\nu_n r)$$

where J_0 and J_1 are Bessel functions and ν_n are the roots of the equation

$$J_1(\nu_n b) = 0 \tag{2.9}$$

(we number them in increasing order). Thereby Q_0v and Q_0w satisfy the boundary conditions for $r = b$. Taking account of the known relationships between J_0 and J_1 for $q_{0n}(t, \xi)$, $p_{0n}(t, \xi)$ we obtain the problem

$$-(\lambda + 2\mu)\nu_n^2 q_{0n} - \nu_n(\lambda + \mu)p_{0n}' + \mu q_{0n}'' = 0 \tag{2.10}$$

$$-\mu\nu_n^2 p_{0n} + (\lambda + 2\mu)p_{0n}'' + (\lambda + \mu)\nu_n q_{0n}' = 0$$

$$\nu_n p_{0n}(t, 0) = q_{0n}'(t, 0),\ p_{0n}(t, 0) = s_{0n}(t)$$

$$\left(s_{0n}(t) = \frac{2}{b^2 J_0^2(\nu_n b)}\int_0^b (\psi_{10}(r, t) - g_0(0, t)) J_0(\nu_n r) r\, dr\right)$$

$$q_{0n}(t, \infty) = p_{0n}(t, \infty) = 0$$

where the prime denotes the derivative with respect to ξ. Its solution has the form

$$q_{0n}(t, \xi) = s_{0n}(t)\left(\frac{\lambda + \mu}{\lambda + 2\mu}\nu_n\xi - \frac{\mu}{\lambda + 2\mu}\right)\exp(-\nu_n\xi)$$

$$p_{0n}(t, \xi) = s_{0n}(t)\left(\frac{\lambda + \mu}{\lambda + 2\mu}\nu_n\xi + 1\right)\exp(-\nu_n\xi)$$

It is known that $\nu_0 = 0$. Consequently, the equality $s_{00}(t) = 0$ should be satisfied to satisfy the condition (2.8), which will enable the boundary value of the function $g_0(z, t)$ to be found for $z = 0$

$$g_0(0, t) = \frac{2}{b^2}\int_0^b \psi_{10}(r, t) r\, dr \equiv g_0^0(t)$$

Thereby the functions Q_0v and Q_0w are defined completely and turn out to be exponentially decreasing as $\xi \to \infty$.

The function $Q_0\theta(r, \xi, t)$ is a solution of the problem

$$L_0Q_0\theta + \partial^2 Q_0\theta/\partial\xi^2 = 0$$

$$r = b,\ \partial Q_0\theta/\partial r = 0;\ \xi = 0,\ Q_0\theta = \mu_{10}(r, t) - \alpha_0(0, t)$$

where the unknown function $\alpha_0(0, t)$ occurs in the boundary condition. We obtain by the method of separation of variables

$$Q_0\theta = \sum_{n=0}^{\infty} d_{0n}(t)\exp(-\nu_n\xi) J_0(\nu_n r)$$

$$d_{0n}(t) = \frac{2}{b^2 J_0^2(\nu_n b)}\int_0^b (\mu_{10}(r, t) - \alpha_0(0, t)) J_0(\nu_n r) r\, dr$$

Taking (2.8) into account we arrive at the equation $d_{00}(t) = 0$, which enables us to determine the boundary value of the function $\alpha_0(z, t)$ for $z = 0$:

$$\alpha_0(0, t) = \frac{2}{b^2}\int_0^b \mu_{10}(r, t)\, r\, dr \equiv \alpha_0^0(t)$$

The function $Q_0\theta$ is thereby defined completely and has, like Q_0v and Q_0w, an exponential estimate in the variable ξ. We note that the convergence of the series for Q_0v, $Q_0w, Q_0\theta$ and the possibility of their double term-by-term differentiation follows from requirements 1°—3°.

We will seek the functions $Q_iv(r, \xi, t)$ and $Q_iw(r, \xi, t)$ for $i \geqslant 1$ again in the form

$$Q_iv = \sum_{n=0}^{\infty} q_{in}(t, \xi) J_1(\nu_n r), \quad Q_iw = \sum_{n=0}^{\infty} p_{in}(t, \xi) J_0(\nu_n r)$$

and for q_{in}, p_{in} we obtain a problem analogous to (2.10) with non-zero right-hand sides in the equations and inhomogeneous boundary conditions for $\xi = 0$. By using (2.8) we here find the boundary values for the functions $g_i(z, t)$ at $z = 0$: $g_i(0, t) = g_i^0(t)$.

For $i \geqslant 1$ the functions $Q_i\theta(r, \xi, t)$ are determined in the same way as $Q_0\theta$. We here find the boundary values for $\alpha_i(z, t)$: $\alpha_i(0, t) = \alpha_i^0(t)$.

The BF $Q_i^*y(r, \xi_*, t)$ $(i = 0, 1, \ldots)$ serve to satisfy the boundary condition for $z = a$ jointly with the regular part of the asymptotic form. They are constructed in the same way as $Q_iy(r, \xi, t)$ and have an exponential estimate in the boundary layer variable ξ_*. Here the boundary values are determined for the functions $g_i(z, t)$ and $\alpha_i(z, t)$ for $z = a : g_i(a, t) = g_i^a(t)$, $\alpha_i(a, t) = \alpha_i^a(t)$.

2.3. *The functions* $g_i(z, t)$. Ordinary differential equations of the type (2.5) were obtained for $g_i(z, t)$ in Sect.2.1, while the boundary values $g_i(0, t) = g_i^0(t)$ and $g_i(a, t) = g_i^a(t)$ were determined in Sect.2.2 for the construction of the Q- and Q^*-functions. Therefore, the functions $g_i(z, t)$ are defined uniquely as the solution of equations of the type (2.5) with the boundary conditions obtained.

Parabolic equations of the type (2.7) were obtained for the functions $\alpha_i(z, t)$ in Sect. 2.1. Consequently, for a unique determination of α_i it is still required to give an initial condition in addition to the boundary values found in Sect.2.2. It will be found during the construction of the Π-, P- and P^*-functions.

2.4. *The boundary layer in the neighbourhood of the initial time.* The function $\Pi y(r, z, \tau, \varepsilon)$ serves to satisfy a given initial condition in combination with the regular part of the asymptotic form. Moreover, the function $\Pi y(r, z, \tau, \varepsilon)$ should be a BF in the variable τ, i.e.,

$$\Pi y(r, z, \tau, \varepsilon) \to 0 \text{ as } \tau \to \infty \tag{2.11}$$

For $\Pi_0v(r, z, \tau)$ and $\Pi_0w(r, z, \tau)$ we obtain problems analogous to those examined in Sect. 2.1 for $\bar{v}_0$ and $\bar{w}_0$. Consequently, $\Pi_0v = 0$, $\Pi_0w = \pi_0(z, \tau)$ is an arbitrary function of the variables z and τ. We find analogously $\Pi_1v = 0$, $\Pi_1w = \pi_1(z, \tau)$ is an arbitrary function of z and τ.

For $\Pi_0\theta(r, z, \tau)$ we have the problem

$$L_0\Pi_0\theta - \varkappa^{-1}\frac{\partial\Pi_0\theta}{\partial\tau} = \eta\frac{\partial^2\pi_0}{\partial z\partial\tau} \tag{2.12}$$
$$r = b,\ \partial\Pi_0\theta/\partial r = 0;\ \tau = 0,\ \Pi_0\theta = \chi_0(r, z) - \alpha_0(z, 0)$$

The second derivatives of the as yet unknown function $\pi_0(z, \tau)$ is on the right-hand side of the equation, while $\alpha_0(z, 0)$, the unknown initial value of the function $\alpha_0(z, t)$ is in the initial condition. We find by separation of variables

$$\Pi_0\theta = \sum_{n=0}^{\infty} b_{0n}(z)\exp(-\varkappa\nu_n^2\tau) J_0(\nu_n r) - \eta\varkappa\frac{\partial\pi_0(z, \tau)}{\partial z} \tag{2.13}$$
$$b_{0n}(z) = \frac{2}{b^2 J_0^2(\nu_n b)}\int_0^b \left(\chi_0(r, z) + \eta\varkappa\frac{\partial\pi_0(z, 0)}{\partial z} - \alpha_0(z, 0)\right) J_0(\nu_n r)\, r\, dr$$

where ν_n are the roots of (2.9). Taking account of (2.11), we arrive at the equation $b_{00}(z) = 0$. It yields a connection between the as yet unknown functions $\alpha_0(z, t)$ and $\partial\pi_0(z, t)/\partial z$ for $t = 0$ and $\tau = 0$:

$$\alpha_0(z, 0) - \eta\varkappa\frac{\partial\pi_0(z, 0)}{\partial z} = \frac{2}{b^2}\int_0^b \chi_0(r, z)\, r\, dr \equiv \gamma_0(z) \tag{2.14}$$

Using (2.14), we determine $b_{0n}(z)$ completely. But for the final determination of $\Pi_0\theta$ the function $\pi_0(z, \tau)$ must still be found.

For $\Pi_2 v(r, z, \tau)$ we obtain the problem

$$(\lambda + 2\mu)(L_0\Pi_2 v - r^{-2}\Pi_2 v) = -\beta \sum_{n=1}^{\infty} \nu_n b_{0n}(z) \exp(-\varkappa\nu_n^2\tau) J_1(\nu_n r), \; r = b, \; \Pi_2 v = 0 \tag{2.15}$$

Its solution is

$$\Pi_2 v = \frac{\beta}{\lambda + 2\mu} \sum_{n=1}^{\infty} \frac{b_{0n}(z)}{\nu_n} \exp(-\varkappa\nu_n^2\tau) J_1(\nu_n r)$$

The function $\Pi_2 w(r, z, \tau)$ is the solution of the problem

$$\mu L_0 \Pi_2 w = -(\lambda + 2\mu)\partial^2\pi_0/\partial z^2; \; r = b, \; \partial\Pi_2 w/\partial r = 0 \tag{2.16}$$

We obtain an equation for $\pi_0(z, \tau) : \partial^2\pi_0/\partial z^2 = 0$ from the condition for problem (2.16) to be solvable, consequently $\Pi_2 w = \pi_2(z, \tau)$ is an arbitrary function of z and τ.

For $\Pi_i v(r, z, \tau)$ for $i \geqslant 3$ we obtain a problem of the type (2.15) which has a unique solution.

For $\Pi_i w(r, z, \tau)$ and $\Pi_{i-2}\theta(r, z, \tau)$ for $i \geqslant 3$ we have problems analogous to (2.16) and (2.12). The function $\Pi_i w$ is determined, apart from the arbitrary function $\pi_i(z, \tau)$, while the unknown function $\partial\pi_{i-2}(z, \tau)/\partial z$ will occur in the expression for $\Pi_{i-2}\theta$. From the condition for the problem for $\Pi_i w$ to be solvable we obtain an ordinary differential equation for $\pi_{i-2}(z, \tau)$ of the form

$$\partial^2\pi_{i-2}/\partial z^2 = l_{i-2}(z, \tau) \tag{2.17}$$

where $l_{i-2}(z, \tau)$ is a known function that has an exponential estimate in τ, while for $\Pi_{i-2}\theta$ we find a relationship of the type (2.14) between $\alpha_{i-2}(z, t)$ and $\partial\pi_{i-2}(z, \tau)/\partial z$ from condition (2.11) for $t = 0$ and $\tau = 0$.

Therefore, the Π-functions can be determined uniquely only after the functions $\pi_i(z, \tau)$ have been found. So far, differential equations of the type (2.17) have been found for them. When constructing angular BF the $\pi_i(z, \tau)$ will be determined completely.

2.5. Angular boundary layer. The functions $\pi_i(z, \tau)$. The functions $Pv(r, \xi, \tau, \varepsilon)$ and $Pw(r, \xi, \tau, \varepsilon)$ are to eliminate residuals introduced by the BF $\Pi v(r, z, \tau, \varepsilon)$ and $\Pi w(r, z, \tau, \varepsilon)$ in the boundary condition at $z = 0$; the function $P\theta(r, \xi, \tau, \varepsilon)$ is to eliminate residuals introduced by BF $\Pi\theta(r, z, \tau, \varepsilon)$ in the boundary condition at $z = 0$ and by the BF $Q\theta(r, \xi, t, \varepsilon)$ in the initial condition at $t = 0$. Moreover, the P-functions should be BF in the variables ξ and τ, i.e., angular BF:

$$Py(r, \xi, \tau, \varepsilon) \to 0 \quad \text{as } \xi + \tau \to \infty \tag{2.18}$$

An analogous system is obtained for $P^*y(r, \xi_*, \tau, \varepsilon)$.

For $P_0 v(r, \xi, \tau)$ and $P_0 w(r, \xi, \tau)$ we have the same problem as in Sect.2.2 for $Q_0 v, Q_0 w$, it is just necessary to replace the last boundary condition by $P_0 w = -\pi_0(0, \tau)$ and the parameter t by τ. Solving this problem taking (2.18) into account, we obtain $\pi_0(0, \tau) = 0$ and $P_0 v = P_0 w = 0$.

In the same manner we find $\pi_0(a, \tau) = 0, P_0^* v(r, \xi_*, \tau) = P_0^* w(r, \xi_*, \tau) = 0$.

The equation $\partial^2\pi_0/\partial z^2 = 0$ was obtained in Sect.2.4 for $\pi_0(z, \tau)$ and now the boundary conditions $\pi_0(0, \tau) = \pi_0(a, \tau) = 0$ were determined for the construction of the angular BF. The problem for π_0 obviously has just the trivial solution $\pi_0(z, \tau) = 0$.

The function $\Pi_0 w = 0$ and $\Pi_0\theta$ are thereby determined completely, where $\Pi_0\theta$ has an exponential estimate in τ (see (2.13)), and we find the initial value for the functions $\alpha_0(z, t)$: $\alpha_0(z, 0) = \gamma_0(z) \equiv \alpha_{00}(z)$ from (2.14).

Similarly

$$\pi_1(z, \tau) = 0, \; P_1 v = P_1 w = P_1^* v = P_1^* w = 0, \; \alpha_1(z, 0) = \alpha_{10}(z)$$

For $P_0\theta(r, \xi, \tau)$ we obtain the problem

$$L_0 P_0\theta + \frac{\partial^2 P_0\theta}{\partial\xi^2} - \varkappa^{-1}\frac{\partial P_0\theta}{\partial\tau} = 0$$

$$r = b, \quad \frac{\partial P_0\theta}{\partial r} = 0; \quad \xi = 0, \quad P_0\theta = -\sum_{n=1}^{\infty} b_{0n}(0) \times \exp(-\varkappa\nu_n^2\tau) J_0(\nu_n r)$$

$$\tau = 0, \quad P_0\theta = -\sum_{n=1}^{\infty} d_{0n}(0)\exp(-\nu_n\xi) J_0(\nu_n r)$$

By virtue of requirement 2° $b_{0n}(0) = d_{0n}(0)$, i.e., the conditions for matching the initial and boundary value are satisfied for the function $P_0\theta$. The solution of the problem is found by separation of variables and has an exponential estimate in the variables ξ and τ.

For $P_i v(r, \xi, \tau)$ and $P_i w(r, \xi, \tau)$ for $i \geqslant 2$ we obtain a problem with non-zero right-hand sides in the equations and inhomogeneous boundary conditions for $\xi = 0$ and $i \geqslant 3$. This problem can also be solved by separation of variables. The boundary value is here determined for $\pi_i(z, \tau)$ for $z = 0$. Similarly, we find $P_i^* v(r, \xi_*, \tau)$, $P_i^* w(r, \xi_*, \tau)$ and $\pi_i(a, \tau)$ for $i \geqslant 2$. Then we solve the equation for $\pi_i(z, \tau)$ with the boundary conditions found and we thereby determine $\Pi_i w$ and $\Pi_i \theta$ completely, and we also find the initial value for the functions $\alpha_i(z, t)$: $\alpha_i(z, 0) = \alpha_{i0}(z)$. Let us note that π_i, and therefore, also $\Pi_i w$ and $\Pi_i \theta$ have an exponential estimate in the boundary layer variable τ since they are solutions of (2.17) with the boundary values found that have an exponential estimate in τ.

For $P_i \theta(r, \xi, \tau)$ for $i \geqslant 1$ we obtain a problem with non-zero right-hand side in the equations and inhomogeneous boundary condition for $r = b$ for $i \geqslant 2$. The boundary and initial conditions are matched for $\xi = 0$ and $\tau = 0$ by virtue of requirement 2°. The solution of the problem can be found by separation of variables and has an exponential estimate in the variables ξ and τ.

The functions $P_i^* \theta(r, \xi_*, \tau)\,(i = 0, 1, \ldots)$ are constructed in an analogous way.

2.6. The functions $\alpha_i(z, t)$. Equations of the type (2.7) were obtained in Sect.2.1 for the functions $\alpha_i(z, t)$ and initial and boundary values were obtained in Sects.2.2 and 2.5 for the construction of BF. It can be shown that they are matched by virtue of requirement 2°, i.e., $\alpha_i^0(0) = \alpha_{i0}(0), \alpha_i^a(0) = \alpha_{i0}(a)$.

We will introduce still another condition.

4°. Let (2.7) with the additional conditions

$$\alpha_0(z, 0) = \alpha_{00}(z),\ \alpha_0(0, t) = \alpha_0^0(t),\ \alpha_0(a, t) = \alpha_0^a(t)$$

have a solution.

The functions $\alpha_i(z, t)$ for $i \geqslant 1$ are later determined successively as solutions of linear equations of the type (2.7) with the additional conditions found above.

Thus, the method described enables us to determine terms of the expansion (2.1) to any number n.

3. Fundamental result. The fundamental result can be formulated as follows. We let $Y_n(r, z, t, \varepsilon)$ denote the n-th partial sum of series (2.1).

Theorem 1. Under conditions 1° – 4° the function $Y_n(r, z, t, \varepsilon)$ satisfies system (1.3) and the additional conditions (1.4) to an accuracy $O(\varepsilon^{n+1})$.

The assertion of the theorem follows directly from the very method of constructing series (2.1).

We note two essential statements associated with the asymptotic form constructed.

1) Finding terms of the asymptotic form (2.1) reduces to solving simpler problems than the initial problem (1.3), (1.4). The regular terms of the asymptotic form were determined by using ordinary differential equations of the type (2.3), (2.5) whose solutions are found in an elementary way in explicit form, and parabolic equations of the type (2.7). If the thermal sources H depend linearly on the temperature, the solution of (2.7) is also found in explicit form. Explicit representations in the form of series are found for the BF by separation of variables.

2) In practical computations the original system of equations is replaced by a simpler shortened system. Which terms of the equations can be discarded? At first glance, it can be shown that the derivatives with respect to ρ can be neglected in the equations for the axial displacement w and the temperature θ for a slender rod with slight heat transfer on its side surface and the following shortened equations (the one-dimensional model) can be considered

$$(\lambda + 2\mu)\frac{\partial^2 w}{\partial z^2} + f = 0, \quad \frac{\partial^2 \theta}{\partial z^2} - \varkappa^{-1}\theta^{\cdot} - \eta\frac{\partial w^{\cdot}}{\partial z} = -\varkappa^{-1}H \tag{3.1}$$

However, an asymptotic analysis shows that this is not so. The equations for the principal terms of the asymptotic form $\bar{w}_0 = g_0(z, t)$ and $\bar{\theta}_0 = \alpha_0(z, t)$ have the forms of (2.5) and (2.7), i.e., differ from (3.1) by additional components showing the need to take account of the weak heat transfer even in the zero approximation (the term $-2b^{-1}A\varkappa\alpha_0$ in (2.7)), and small radial displacements on the side surface (the terms $-2b^{-1}(\lambda + \mu)(\lambda + 2\mu)^{-1}\,\partial\varphi_0/\partial z$ in (2.5) and $-2b^{-1}\eta\varkappa\varphi_0^{\cdot}$ in (2.7)). Therefore, an asymptotic analysis affords the possibility of an exact answer to the question of what one-dimensional model is correct.

4. Theorem of existence. Let us put

$$v^- = v - V_{n+2}(r, z, t, \varepsilon) - (\varepsilon\varphi(z, t, \varepsilon) - V_{n+2}(b, z, t, \varepsilon))$$
$$w^- = w - W_{n+2}(r, z, t, \varepsilon) - ((\psi_1(r, t, \varepsilon) - W_{n+2}(r, 0, t, \varepsilon))\times (a - z)/a + (\psi_2(r, t, \varepsilon) - W_{n+2}(r, a, t, \varepsilon))\,z/a)$$
$$\theta^- = \theta - \Theta_{n+2}(r, z, t, \varepsilon)$$

where $V_{n+2}, W_{n+2}, \Theta_{n+2}$ are partial sums of series (2.1) for v, w, θ, respectively. For v^-, w^-, θ^- we obtain the problem

$$L_1 y^- = \varepsilon^2 \beta \partial\theta^-/\partial r + m_1(r, z, t, \varepsilon)$$
$$L_2 y^- = \varepsilon^3 \beta \partial\theta^-/\partial z + m_2(r, z, t, \varepsilon) \tag{4.1}$$

$$L_3 y^- = h(\theta^-, r, z, t, \varepsilon), \quad (r, z, t) \in G \tag{4.2}$$

$$r = b, \; v^- = 0, \; \partial w^-/\partial r = 0$$
$$z = 0, \; \partial w^-/\partial r + \varepsilon \partial v^-/\partial z = 0, \; w^- = 0 \tag{4.3}$$
$$z = a, \; \partial w^-/\partial r + \varepsilon \partial v^-/\partial z = 0, \; w^- = 0$$

$$r = b, \; \partial\theta^-/\partial r + A\varepsilon^2\theta = O(\varepsilon^{n+3})$$
$$z = 0, \; \theta^- = O(\varepsilon^{n+3}); \; z = a, \; \theta^- = O(\varepsilon^{n+3}) \tag{4.4}$$
$$t = 0, \; \theta^- = O(\varepsilon^{n+3})$$
$$(h(\theta^-, r, z, t, \varepsilon) = -\varepsilon^2 \varkappa^{-1} H(\Theta_{n+2} + \theta^-, r, z, t) - L_3(y - y^-))$$

where $m_i \; (i = 1, 2)$ are known functions: $m_i = O(\varepsilon^{n+3})$ in G.

We note that the introduction of the functions v^-, w^-, θ^- in exactly the same way as this has been done would enable us to obtain homogeneous boundary conditions for v^-, w^-.

One more condition is needed to prove the existence of a solution of problem (4.1)-(4.4).

5°. Let $m_1(0, z, t, \varepsilon) = 0$.

The sufficient condition for 5° is satisfaction of the following equalities: $\varphi_{i+2}(z, t) = 0$, $F_{i+1}(0, z, t) = 0$ for $i > n$.

Let us consider the equation obtained from (4.2) if we replace the terms

$$\left\{-\varepsilon\eta \frac{1}{r}\frac{\partial}{dr}(rv^{\cdot -}) - \varepsilon^2\eta \frac{\partial w^{\cdot -}}{\partial z}\right\} \text{ by } \left\{-\frac{\varepsilon^3\beta\eta}{\lambda + 2\mu}\theta^{\cdot -} + M(r, z, t, \varepsilon)\right\}$$

where M is a certain known function. The existence and uniqueness of a solution can be proved for this equation with the additional conditions (4.4) (see /6/). We expand this solution in series

$$\theta^- = \Sigma\theta_{nk} J_0(\nu_n r) \cos \pi k a^{-1} z$$

The summation here and below is over n and k between zero and infinity; the quantities with subscript nk are functions of t and ε and ν_n are the roots of (2.9). Let us substitute this series into (4.1). The solution of problem (4.1) and (4.3) is sought by separation of variation

$$v^- = \Sigma v_{nk} J_1(\nu_n r) \cos \pi k a^{-1} z$$
$$w^- = \Sigma w_{nk} J_0(\nu_n r) \sin \pi k a^{-1} z \tag{4.5}$$

We also expand the functions m_1, m_2 in series

$$m_1 = \Sigma m_{1nk} J_1(\nu_n r) \cos \pi k a^{-1} z, \quad m_2 = \Sigma m_{2nk} J_0(\nu_n r) \sin \pi k a^{-1} z$$

We note that the series for m_1 converges uniformly in G by virtue of requirement 5°.

Substituting these representations into (4.1), we obtain a system of two algebraic equations for each pair of coefficients v_{nk}, w_{nk} and by solving it we express v_{nk}, w_{nk} in terms of $\theta_{nk}, m_{1nk}, m_{2nk}$. The solution of (4.5) is thereby found. Now it can be seen that

$$-\varepsilon\eta \frac{1}{r}\frac{\partial}{\partial r}(rv^{\cdot -}) - \varepsilon^2\eta \frac{\partial w^{\cdot -}}{\partial z} = -\frac{\varepsilon^3\beta\eta}{\lambda + 2\mu}\theta^{\cdot -} + M(r, z, t, \varepsilon)$$

$$M = \Sigma \frac{\nu_n \dot{m}_{1nk} + \varepsilon\pi k a^{-1} \dot{m}_{2nk}}{(\lambda + 2\mu)(\nu_n^2 + (\varepsilon\pi k a^{-1})^2)} = O(\varepsilon^{n+3})$$

Consequently, as /6/ also, an estimate is obtained for θ^-

$$\max_{\bar{G}} |\theta^-| = O(\varepsilon^{n+1})$$

Analogous estimates follow in an elementary way for v^-, w^-.

Therefore the following is proved.

Theorem 2. If conditions 1°–5° are satisfied, then for sufficiently small ε problem (1.3) and (1.4) has the solution $y(r, z, t, \varepsilon)$ for which the series (2.1) is asymptotic in the domain $\bar{G}$, i.e.,

$$\max_{\bar{G}} |y - Y_n| = O(\varepsilon^{n+1})$$

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